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Generic quotient varieties

AKIHIKO GYOJA

Introduction. The purpose of this note is to review [**R1**, Theorem 2].

Theorem 0.1. *Let k be an algebraically closed field, X an irreducible algebraic variety over k, and G an algebraic group acting on X. Then there exists an open dense $X_0 \subset X$, a variety W_0, and a morphism $\phi : X_0 \to W_0$ such that*

(1) $GX_0 = X_0$,

(2) every fibre of ϕ is precisely a single G-orbit,

(3) $\phi : X_0 \to W_0$ is smooth,

(4) X_0 and W_0 are non-singular,

(5) $\phi^* : k(W_0) \xrightarrow{\simeq} k(X_0)^G$ $(= k(X)^G)$, and

(6) $\phi^* : k[W_0] \xrightarrow{\simeq} k[X_0]^G$.

*(Cf. (0.4) for notation.) In particular, $\phi : X_0 \to W_0$ is a geometric quotient in the sense of D.Mumford [**Mu**, p.4].*

This theorem can be used in the theory of prehomogeneous vector spaces as follows. Put

$$m := \max\{\dim Gx \mid x \in X\}.$$

Then this maximum is attained by x's belonging to a dense subset of X. From the above theorem, we get the following results concerning m.

Corollary 0.2. *Let notation be as in (0.1). Then* $\dim X - m = \dim W_0 = \operatorname{tr.deg}_k k(X)^G$.

Corollary 0.3. *Let notation be as in (0.1). The following conditions are equivalent.*

(1) X *has an open dense* G*-orbit.*

(2) $\mathrm{tr.deg}_k\, k(X)^G = 0$.

(3) $k(X)^G = k$.

0.4. Convention and Notation. We fix an algebraically closed field k, and we always assume that an algebraic variety is defined over k unless otherwise stated. We identify a (k-)variety, say X, with the set of its rational points $X(k)$. We denote by $k[X]$ (resp. $k(X)$) the ring of regular functions (resp. the filed of rational functions if X is irreducible) on X. For a group Γ acting on a set A, $A^\Gamma := \{a \in A \mid \gamma a = a$ for all $\gamma \in \Gamma\}$.

§1.

1.1. Flatness. The concept of 'flatness' plays an important role in the algebraic geometry [**EGA**]. A concise account can be found in [**Mi**, Chapter 1]. We recall two lemmas from [**EGA**].

Lemma 1.2. *[***EGA***, (IV, 6.9.1)]. Let* Y *be a locally noetherian (I, 2.7.1), integral scheme (I, 2.1.8), and* $u : X \to Y$ *a morphism of finite type (I, 6.3.2). Then there exists an open dense* $U \subset Y$ *such that* $u : u^{-1}(U) \to U$ *is flat.*

Lemma 1.3. *[***EGA***, (IV, 2.4.6)]. Let* $f : X \to Y$ *be a flat morphism of locally of finite presentation (I, 6.2.1). Then* f *is universally open (IV, 2.4.2).*

1.4. Hilbert scheme.[**Mu**, pp.21–22]. It is known that there exist

(1) a locally noetherian $\mathbb{Z}$-scheme $Hilb_{\mathbb{P}^n_{\mathbb{Z}}}$ whose connected components are projective over $\mathrm{Spec}(\mathbb{Z})$, and

(2) a closed $\mathbb{Z}$-subscheme $W_{\mathbb{Z}} \subset \mathbb{P}^n_{\mathbb{Z}} \times Hilb_{\mathbb{P}^n_{\mathbb{Z}}}$ flat over $Hilb_{\mathbb{P}^n_{\mathbb{Z}}}$,

such that for

(3) any locally noetherian $\mathbb{Z}$-scheme $S_{\mathbb{Z}}$, and

(4) any closed $\mathbb{Z}$-scheme $Z_{\mathbb{Z}} \subset \mathbb{P}^n_{\mathbb{Z}} \times S_{\mathbb{Z}}$, flat over $S_{\mathbb{Z}}$,

there is a unique morphism $f_{\mathbb{Z}} : S_{\mathbb{Z}} \to Hilb_{\mathbb{P}^n_{\mathbb{Z}}}$ such that $Z_{\mathbb{Z}} = (1_{\mathbb{P}^n_{\mathbb{Z}}} \times f_{\mathbb{Z}})^*(W_{\mathbb{Z}})$.

1.5. If we put $Hilb_{\mathbb{P}^n_k} := Hilb_{\mathbb{P}^n_{\mathbb{Z}}} \times_{\mathrm{Spec}(\mathbb{Z})} \mathrm{Spec}(k)$ and $W_k := W_{\mathbb{Z}} \times_{\mathrm{Spec}(\mathbb{Z})} \mathrm{Spec}(k)$, we may replace $\mathbb{Z}$ with k everywhere in (1.3).

§2.

We start this section with the following simple lemma.

Lemma 2.1. *Let $f : X \to Y$ be a morphism between algebraic varieties, and $Z \subset X$ a constructible subset (i.e., a finite disjoint union of locally closed subsets with repsect to the Zariski topology.) We further assume that Z is irreducible and $\overline{f(Z)} = Y$. Then*

$$Y_0 := \{y \in Y \mid \overline{f^{-1}(y) \cap Z} = f^{-1}(y) \cap \overline{Z}\} \tag{1}$$

contains an open dense subset of Y. (Let $Y^\sharp$ denote the largest open subset of Y contained in Y_0.)

Proof. Let Z_0 be the largest subset of Z which is open in $\overline{Z}$. Since Z is constructible, Z_0 is open dense in $\overline{Z}$. Let $\{Z_i\}_{i\in I}$ (resp. $\{Z'_j\}_{j\in J}$) be all the irreducible components of $\overline{Z} \setminus Z_0$ such that $\overline{f(Z_i)} = Y$ (resp. $\overline{f(Z_i)} \subsetneqq Y$). Let Y_1 be the totality of $y \in Y$ such that

$$\dim(f^{-1}(y) \cap Z_i) = \dim Z_i - \dim Y \quad \text{for all } i \in I \sqcup \{0\}, \text{ and} \tag{2}$$

$$f^{-1}(y) \cap Z'_j = \phi \quad \text{for all } j \in J. \tag{3}$$

Then Y_1 is a constructible dense subset of Y. (The condition (2) gives an open subset, while (3) gives a constructible one in general.) For $y \in Y_1$, we have

$$(4) \qquad f^{-1}(y) \cap Z \subset \overline{f^{-1}(y) \cap Z} \subset f^{-1}(y) \cap \overline{Z} = \bigcup_{i \in I \sqcup \{0\}} (f^{-1}(y) \cap Z_i).$$

by (3). Especially,

$$\begin{aligned} \dim(f^{-1}(y) \cap \overline{Z}) &= \max_{i \in I \sqcup \{0\}} \dim(f^{-1}(y) \cap Z_i) \\ &= \max_{i \in I \sqcup \{0\}} (\dim Z_i - \dim Y) \text{ by (2)} \\ &= \dim Z_0 - \dim Y = \dim \overline{Z} - \dim Y. \end{aligned}$$

(Indeed, $\dim Z_i < \dim Z_0$ for all $i \in I$.) In other words, the fibres of $f : \overline{Z} \to Y$ attain the minimum dimension at $y \in Y_1$. Hence all the irreducible components of $f^{-1}(y) \cap \overline{Z}$ are of the same dimension $\dim \overline{Z} - \dim Y$. (See [**EGA**, (IV, 13.2)] for the related generality.) Since

$$\dim(f^{-1}(y) \cap Z_i) = \dim Z_i - \dim Y < \dim \overline{Z} - \dim Y$$

for $i \in I$, $f^{-1}(y) \cap Z_i$ ($i \in I$) are nowhere dense in $f^{-1}(y) \cap \overline{Z}$, and consequently (4) yields that $f^{-1}(y) \cap Z_0$ ($\subset f^{-1}(y) \cap Z$) is dense in $f^{-1}(y) \cap \overline{Z}$. Hence $\overline{f^{-1}(y) \cap Z} = f^{-1}(y) \cap \overline{Z}$, i.e., $Y_1 \subset Y_0$. Since Y_1 is constructible and dense in Y, we get the desired result. ∎

2.2. Let k, G, X be as in the introduction, and $\overline{X}$ an irreducible projective variety containing X as an open dense subset. (Such $\overline{X}$ exists [**N**], but possibly the G-action on X can not be extended to $\overline{X}$. See [**S**] for equivariant completions.) Let Z be the Zariski closure of $\{(gx, x) \mid x \in X, g \in G\}$ in $\overline{X} \times X$, and $\pi : Z \to X$ the second projection. Intuitively, $\pi : \pi^{-1}(X^\sharp) \to X^\sharp$ is the family of orbit closures $\overline{Gx}$ in $\overline{X}$

parametrized by $x \in X^\sharp$, where $X^\sharp$ is defined as in (2.1) using Z and $\pi : \overline{X} \times X \to X$ in place of Z and $f : X \to Y$. Let X_0 be the largest open (dense) subset of $X^\sharp$ such that $\pi : \pi^{-1}(X_0) \to X_0$ is flat (cf. (1.2)). Then applying (1.5) to $S = X_0$ and $Z = \pi^{-1}(X_0)$ $(\subset \overline{X} \times X_0 \subset \mathbb{P}^n_k \times X_0$ for some $n)$, we get a morphism $f : X_0 \to Hilb_{\mathbb{P}^n_k}$ which makes the following diagram cartesian.

$$\begin{array}{ccc} \pi^{-1}(X_0) & \longrightarrow & W \\ \downarrow & & \downarrow \\ X_0 & \xrightarrow{f} & Hilb_{\mathbb{P}^n_k} \end{array}$$

Let Y_1 be the largest open subset of $\overline{f(X_0)}$ such that $f : f^{-1}(Y_1) \to Y_1$ is flat (cf. (1.2)) and surjective. Put $X_1 = f^{-1}(Y_1)$.

Lemma 2.3. *(1) The open dense subset X_1 $(\subset X)$ is preserved by G.*

(2) The fibres of $f : X_1 \to Y_1$ are precisely the G-orbit in X_1.

(3) $f : X_1 \to Y_1$ is universally open.

Proof. (1) is obvious. (3) follows from (1.3). For $x \in X_1$, let $i_x : \mathrm{Spec}(k) \to X_1$ be the corresponding geometric point. Then we get cartesian squares

$$\begin{array}{ccccc} \overline{Gx} & \longrightarrow & \pi^{-1}(X_1) & \longrightarrow & W \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{Spec}(k) & \xrightarrow{i_x} & X_1 & \xrightarrow{f} & Hilb_{\mathbb{P}^n_k} \end{array}$$

Hence for $x, x' \in X_1$,

$$Gx = Gx' \Leftrightarrow \overline{Gx} = \overline{Gx'} \Leftrightarrow f \circ i_x = f \circ i_{x'} \Leftrightarrow f(x) = f(x').$$

(To see the first equivalence, note that Gx is the unique G-orbit which is open in $\overline{Gx} \cap X$. The second equivalence follows from the uniqueness part in (1.4).) ∎

§3.

We need some preliminary from the field theory.

Lemma 3.1. *If L/K is a finitely generated field extension, and M a field such that $K \subset M \subset L$, then M/K is also finitely generated.*

Proof. Let K' be a purely transcendental extension of K, contained in M, and such that the transcendental degree $\mathrm{tr.deg}_K K'$ ($< \mathrm{tr.deg}_K L < +\infty$) is maximal among such extensions. Replacing K' with K, we may assume that M/K is an algebraic extension.

Let N be a purely transcendental extension of K, contained in L, and such that $\mathrm{tr.deg}_K N$ is maximal among such extensions. Then L/N is an algebraic, finitely generated extension, i.e., $[L : N] < +\infty$. On the other hand, M/K is algebraic, N/K is purely transcendental, and hence they are lineary disjoint. Therefore $[M : K] = [MN : N] \leq [L : N] < +\infty$. ∎

3.2. Separably generated extension. ([**W**, p.14]) A finitely generated extension is called *a separably generated extension* if it is a separably algebraic extension of a purely transcendental extension.

Concerning this concept, we need the following easier half of [**W**, Chap.1, Prop.19].

Lemma 3.3. *Let L/K be a finitely generated field extension contained in a fixed algebraically closed field. If $K^{p^{-1}}$ and L are linearly disjoint over K, then L/K is a separably generated extension.*

Proof. (An extract from [**W**].) Let $L = K(a_1, \cdots, a_n)$, and let us prove the lemma by induction on n. Let $I := \{f \in K[x_1, \cdots, x_n] \mid f(a_1, \cdots, a_n) = 0\}$, where $x_1, \cdots x_n$ are indeterminates. If $I = 0$, then our conclusion holds. If not, let

$P(x_1, \cdots, x_n) \in I \setminus \{0\}$ be a polynomial of minimal degree. Put $P_i := \partial P / \partial x_i$. If $P_1 = \cdots = P_n = 0$, then $P = Q^p$ with some $Q \in K^{p^{-1}}[x_1, \cdots, x_n]$. Then $Q \in \{g \in K^{p^{-1}}[x_1, \cdots, x_n] \mid g(a_1, \cdots, a_n) = 0\}$, and, by the sublemma below (with $K' = K^{p^{-1}}$), we can see that the right hand side is $\{\sum_i \lambda_i g_i \mid \lambda_i \in K^{p^{-1}}, g_i \in I\}$. This is impossible since $\deg Q < \deg P$. Therefore, we may assume that $P_n \neq 0$. Since $\deg P_n < \deg P$, $P_n(a_1, \cdots, a_n) \neq 0$. This means that a_n is separable over $L' := K(a_1, \cdots, a_{n-1})$. Since $K^{p^{-1}}$ and L' are linearly disjoint, L'/K is separably generated extension by the induction hypothesis. Hence $L = L'(a_n)$ is separably generated over K.

Sublemma. *Let L/K and K'/K be field extensions in a fixed algebraically closed field, and assume that L and K' are linearly disjoint over K. Let $a_1, \cdots, a_n \in L$ and $g \in K'[x_1, \cdots, x_n]$, and assume $g(a_1, \cdots, a_n) = 0$. Then there exists $\kappa_i \in K'$ and $g_i \in K[x_1, \cdots, x_n]$ $(1 \leq i \leq N)$ such that $g_i(a_1, \cdots, a_n) = 0$ and $g(x_1, \cdots, x_n) = \sum_i \kappa_i g_i(x_1, \cdots, x_n)$.*

Proof. Let $\{\kappa_\iota\}_\iota$ be a K-linear basis of K'. Then g can be uniquely expressed as $g = \sum_\iota \kappa_\iota g_\iota$ (finite sum) with $g_\iota \in K[x_1, \cdots, x_n]$. Since

(1) $0 = g(a_1, \cdots, a_n) = \sum_\iota \kappa_\iota g_\iota(a_1, \cdots, a_n)$,

(2) $\kappa_\iota \in K'$ are linearly independent over K,

(3) $g_\iota(a_1, \cdots, a_n) \in L$, and

(4) L and K' are linearly disjoint over K,

it follows that $g_\iota(a_1, \cdots, a_n) = 0$. ∎

Lemma 3.4. *Let $f : X \to Y$ be a dominant morphism between irreducible varieties. Then there exists an open dense $U \subset X$ such that $f|U$ is étale (resp. smooth) [**EGA**, (IV, §17)] if and only if $k(X)/k(Y)$ is a separably algebraic extension (resp. a separably generated extension).*

What is necessary for our present purpose is the 'if part' whose proof is an easy exercise. For the 'only if part', see [**Mi**, Chap.1, §3] and [**SGA**, exposé II]. ∎

We also need the following lemma of M.Rosenlicht [**R2**, p.4, ↑ ℓ.8 – p.5, ↓ ℓ.9].

Lemma 3.5. *Let L be a field, G a group of field automorphisms of L, and $K = L^G$ the subfield of L consisting of all elements of L left fixed by each automorphism of G. Then L/K is separably generated.*

Proof. (An extract from [**R2**].) By (3.3), it suffices to show that $K^{p^{-1}}$ and L are linearly disjoint over K, i.e., that if we have a relation $\sum_{i=1}^n \kappa_i \lambda_i^p = 0$, where $\kappa_i \in K$, $\lambda_i \in L$ and where not all κ_i's are 0, then $\lambda_1, \cdots, \lambda_n$ are linearly dependent over K. Clearly we may take $n > 1$. If $\sigma_1, \cdots, \sigma_n \in G$, we have $\sum_{i=1}^n \kappa_i \sigma_j(\lambda_i^p) = 0$ $(j = 1, \cdots, n)$, so $\det(\sigma_j(\lambda_i^p))_{1 \le i,j \le n} = 0$ and hence $\det(\sigma_j(\lambda_i))_{1 \le i,j \le n} = 0$. Let r be the maximal rank that $(\sigma_j(\lambda_i))_{1 \le i,j \le n}$ can assume for $\sigma_1, \cdots, \sigma_n \in G$; then $1 \le r < n$. Reorder λ_i's and choose $\sigma_1, \cdots, \sigma_r \in G$ so that $\det(\sigma_j(\lambda_i))_{1 \le i,j \le r} \ne 0$. Hold $\sigma_1, \cdots, \sigma_r$ fixed, and let $\sigma_{r+1} \in G$ be arbitrary. Then $\det(\sigma_j(\lambda_i))_{1 \le i,j \le r+1} = 0$, so there exist $\mu_1, \cdots, \mu_r \in L$ such that

$$(1)_j \qquad \sigma_j(\lambda_{r+1}) = \sum_{i=1}^r \mu_i \sigma_j(\lambda_i)$$

for all $j = 1, \cdots, r+1$. Here $(1)_{r+1}$ is redundant, and $\mu_1, \cdots, \mu_r$ are uniquely determined only by $(1)_j$, $1 \le j \le r$. Therefore these μ_i's are independent of the choice of σ_{r+1}, and hence we have $\sigma(\lambda_{r+1}) = \sum_{i=1}^r \mu_i \sigma(\lambda_i)$ for any $\sigma \in G$. If $\tau \in G$, we have

$$\sigma(\lambda_{r+1}) = \tau\left(\tau^{-1}\sigma(\lambda_{r+1})\right) = \tau\left(\sum_{i=1}^r \mu_i \cdot \tau^{-1}\sigma(\lambda_i)\right) = \sum_{i=1}^r \tau(\mu_i) \cdot \sigma(\lambda_i).$$

By the uniqueness of $\mu_1, \cdots, \mu_r$, we have $\tau(\mu_i) = \mu_i$, so each $\mu_i \in K = L^G$. Hence any one of $(1)_j$ yields $\lambda_{r+1} = \sum_{i=1}^r \mu_i \lambda_i$ with $\mu_1, \cdots, \mu_r \in K$. Hence $\lambda_1, \cdots, \lambda_n$ are linearly dependent over K. ∎

3.6. *Proof of Theorem 0.1.* Now let us return to (2.3). Put $L = k(X_1)$ and $K = L^G$. By (3.1), K/k is finitely generated. Hence there is an irreducible k-variety W_1 such that $K = k(W_1)$, and we get a G-equivariant dominant rational morphism $\phi : X_1 \to W_1$. (The G-action on W_1 is trivial.) Let X_2 be the locus where ϕ is defined and smooth. Put $\phi(X_2) =: W_2$. Then $W_2 \subset W_1$ is open dense (cf. (3.1) and (3.5)), $\phi : X_2 \to W_2$ is an open mapping (cf. (1.3)), and $GX_2 = X_2$.

By (2.3, (2)), $k(Y_1) \subset k(X_1)^G = k(W_1) = k(W_2)$. Hence we get a dominant rational morphism $\psi : W_2 \to Y_1$. Take open dense $W_3 \subset W_2$ and $Y_3 \subset Y_1$ so that $\psi : W_3 \to Y_3$ is surjective regular morphism. Put $X_3 := \phi^{-1}(W_3)$.

Since $\phi : X_3 \to W_3$ is G-equivariant, each fibre of ϕ is a union of G-orbits. But each fibre of $f = \psi \circ \phi$ is precisely a G-orbit. Hence each fibre of ϕ is also precisely a G-orbit.

Hence all the assertions of (2.3) remain valid when $f : X_1 \to Y_1$ is replaced with $\phi : X_3 \to W_3$. Moreover $\phi : X_3 \to W_3$ is smooth, and $k(W_3) = k(X_3)^G$.

Let W_4 be the non-singular locus of W_3, and put $X_4 := \phi^{-1}(W_4)$. Then all the conditions (0.1, (1)–(5)) are satisfied, and

$$\phi^* k[W_4] \subset k[X_4]^G \subset k(X_4)^G = \phi^* k(W_4).$$

In order to prove (0.1, (6)), let us assume the contrary, i.e., that there exists $\alpha \in k[X_4]^G \setminus \phi^* k[W_4]$. Then $\alpha = \phi^* \beta$ with som $\beta \in k(W_4) \setminus k[W_4]$. Let W_4' be the locus where $1/\beta$ is regular. Since W_4 is a normal variety, $Z := \{w \in W_4' \mid \beta(w)^{-1} = 0\}$ is a non-empty subvariety of a pure codimention one. Take $x_0 \in X_4$ so that $\phi(x_0) \in Z$. Then both $\phi^*\beta$ and $1/\phi^*\beta$ are regular on $\phi^{-1}(W_4)$, and

$$1 = (\phi^*\beta)(x_0) \cdot (\phi^*\beta)(x_0)^{-1} = (\phi^*\beta)(x_0) \times 0.$$

Thus we get a contradition, and get (0.1, (6)). ∎

3.7. Remark. In order to simplify the exposition, we assumed in (0.1) that k is algebraically closed and that X is irreducible, but these assumptions are not essential, and indeed are not assumed in [**R1**].

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